SM212, Undetermined coefficients

PROBLEM: Solve

$$ay'' + by' + cy = f(x). \tag{1}$$

We *assume* that $f(x)$ is of the form $c \cdot p(x) \cdot e^{ax} \cdot \cos(bx)$, or $c \cdot p(x) \cdot e^{ax} \cdot \sin(bx)$, where a, b, c are constants and $p(x)$ is a polynomial.

soln:

- Find the "homogeneous solution" y_h to $ay'' + by' = cy = 0$, $y_h = c_1y_1 + c_2y_2$.
- Compute $f(x)$, $f'(x)$, $f''(x)$, ... Write down the list of all the different terms which arise, ignoring constrant factors:

  $$f_1(x),\ f_2(x),\ ...,\ \ f_k(x).$$

  If any one of these agrees with y_1 or y_2 then multiply them all by x. (If, after this, any of them *still* agrees with y_1 or y_2 then multiply them all again by x.)
- Let y_p be a linear combination of these functions (your "guess"):

  $$y_p = A_1f_1(x) + ... + A_kf_k(x).$$

  This is called the **general form of the particular solution**. The A_i's are called **undetermined coefficients**.
- Plug y_p into (1) and solve for A_1, ..., A_k.
- Let $y = y_h + y_p = y_p + c_1y_1 + c_2y_2$. This is the **general solution** to (1). If there are any initial conditions for (1), solve for then c_1, c_2 now.

Examples

Example 1 *Solve*

$$y'' - y = \cos(2x).$$

- *The characteristic polynomial is* $r^2 - 1 = 0$*, which has* ± 1 *for roots. The "homogeneous solution" is therefore* $y_h = c_1 e^x + c_2 e^{-x}$.
- *We compute* $f(x) = \cos(2x)$, $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$, ... *. They are all linear combinations of*

  $$f_1(x) = \cos(2x), \;\; f_2(x) = \sin(2x).$$

  None of these agrees with $y_1 = e^x$ *or* $y_2 = e^{-x}$*, so we do not multiply by* x.
- *Let* y_p *be a linear combination of these functions:*

  $$y_p = A_1 \cos(2x) + A_2 \sin(2x).$$

- *You can compute both sides of* $y_p'' - y_p = \cos(2x)$*:*

  $$(-4A_1 \cos(2x) - 4A_2 \sin(2x)) - (A_1 \cos(2x) + A_2 \sin(2x)) = \cos(2x).$$

  Equating the coefficients of $\cos(2x)$, $\sin(2x)$ *on both sides gives 2 equations in 2 unknowns:* $-5A_1 = 1$ *and* $-5A_2 = 0$*. Solving, we get* $A_1 = -1/5$ *and* $A_2 = 0$.
- *The general solution:* $y = y_h + y_p = c_1 e^x + c_2 e^{-x} - \frac{1}{5}\cos(2x)$.

Example 2 *Solve*

$$y'' - y = x\cos(2x).$$

- *The characteristic polynomial is* $r^2 - 1 = 0$*, which has* ± 1 *for roots. The "homogeneous solution" is therefore* $y_h = c_1 e^x + c_2 e^{-x}$.
- *We compute* $f(x) = x\cos(2x)$, $f'(x) = \cos(2x) - 2x\sin(2x)$, $f''(x) = -2\sin(2x) - 2\sin(2x) - 2x\cos(2x)$, ... *. They are all linear combinations of*

  $$f_1(x) = \cos(2x), \;\; f_2(x) = \sin(2x), \;\; f_3(x) = x\cos(2x), \;\; .f_4(x) = x\sin(2x).$$

  None of these agrees with $y_1 = e^x$ *or* $y_2 = e^{-x}$*, so we do not multiply by* x.

- *Let y_p be a linear combination of these functions:*

$$y_p = A_1 \cos(2x) + A_2 \sin(2x) + A_3 x \cos(2x) + A_4 x \sin(2x).$$

- *In principle, you can compute both sides of $y_p'' - y_p = x\cos(2x)$ and solve for the A_i's. (Equate coefficients of $x\cos(2x)$ on both sides, equate coefficients of $\cos(2x)$ on both sides, equate coefficients of $x\sin(2x)$ on both sides, and equate coefficients of $\sin(2x)$ on both sides. This gives 4 equations in 4 unknowns, which can be solved.) You will not be asked to solve for the A_i's for a problem this hard.*

Example 3 *Solve*

$$y'' + 4y = x\cos(2x).$$

- *The characteristic polynomial is $r^2 + 4 = 0$, which has $\pm 2i$ for roots. The "homogeneous solution" is therefore $y_h = c_1 \cos(2x) + c_2 \sin(2x)$.*

- *We compute $f(x) = x\cos(2x)$, $f'(x) = \cos(2x) - 2x\sin(2x)$, $f''(x) = -2\sin(2x) - 2\sin(2x) - 2x\cos(2x)$, ... They are all linear combinations of*

$$f_1(x) = \cos(2x),\ f_2(x) = \sin(2x),\ f_3(x) = x\cos(2x),\ .f_4(x) = x\sin(2x).$$

Two of these agree with $y_1 = \cos(2x)$ or $y_2 = \sin(2x)$, so we do multiply by x:

$$f_1(x) = x\cos(2x),\ f_2(x) = x\sin(2x),\ f_3(x) = x^2\cos(2x),\ .f_4(x) = x^2\sin(2x).$$

- *Let y_p be a linear combination of these functions:*

$$y_p = A_1 x\cos(2x) + A_2 x\sin(2x) + A_3 x^2\cos(2x) + A_4 x^2\sin(2x).$$

- *In principle, you can compute both sides of $y_p'' + 4y_p = x\cos(2x)$ and solve for the A_i's. You will not be asked to solve for the A_i's for a problem this hard.*